Mechanics of Materials
Fundamentals of Engineering Review Course

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OUTLINE:

I. AXIAL LOADS — STRESS AND STRAIN
II. LINEAR STRESS — STRAIN LAW (HOOKE'S LAW)
III. TORSION OF MEMBERS WITH CIRCULAR CROSS SECTION
IV. SHEAR AND BENDING MOMENT IN BEAMS
V. FLEXURAL STRESSES IN BEAMS
VI. SHEARING STRESSES IN BEAMS
VII. DEFLECTION OF BEAMS BY SUPERPOSITION
VIII. STATICALLY INDETERMINATE BEAMS BY SUPERPOSITION
IX. EULER COLUMN EQUATION

INTRODUCTION:

THE PRINCIPAL PROBLEM OF STRENGTH OF MATERIALS IS THE INVESTIGATION OF INTERNAL STRESSES AND STRAINS OF A SLENDER STRUCTURAL MEMBER WHEN THE MEMBER IS SUBJECTED TO EXTERNAL LOADS. BASIC TO THE STUDY IS THE ANALYSIS OF EXTERNAL LOADS WHICH IS COVERED IN THE STUDY OF EQUILIBRIUM IN STATICS. WE WILL ASSUME THAT IF A MEMBER AS A WHOLE IS IN EQUILIBRIUM THEN ANY PART OF IT MUST ALSO BE IN EQUILIBRIUM. THUS AT ANY SECTION WITHIN A MEMBER THERE MUST BE AN INTERNAL FORCE SYSTEM DEVELOPED WHICH MAINTAINS THE

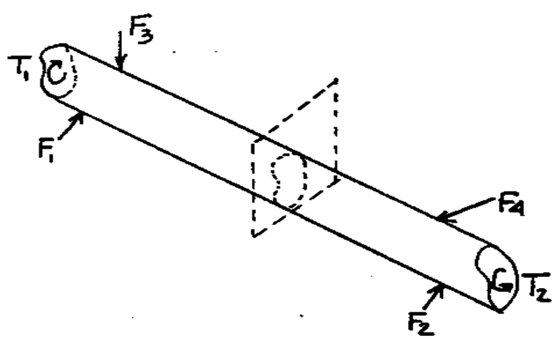
EQUILIBRIUM OF THE PARTS TO EITHER SIDE OF THE SECTION. THE FIGURE BELOW ILLUSTRATES THIS PRINCIPAL WITH THE FORCE SYSTEM SHOWN IN TERMS OF COMPONENTS AT THE CENTROID OF THE SECTION. IF THE X-COORDINATE IS TAKEN PERPINDICULAR TO THE PLANE OF THE SECTION, THE COMPONENTS MAY BE GIVEN THE FOLLOWING INTERPRETATIONS:

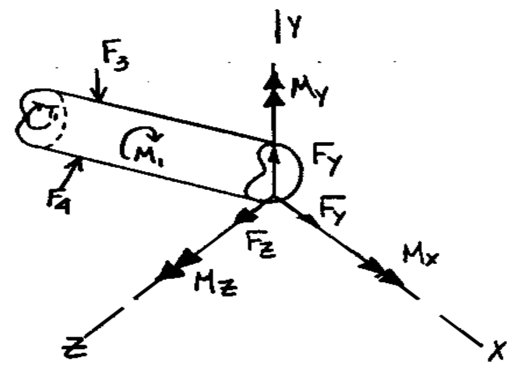
$F_x = P$ (AXIAL FORCE)

$M_x = T$ (TORQUE)

$F_y, F_z = V_y, V_z$ (SHEAR FORCE)

M_y, M_z = BENDING MOMENTS



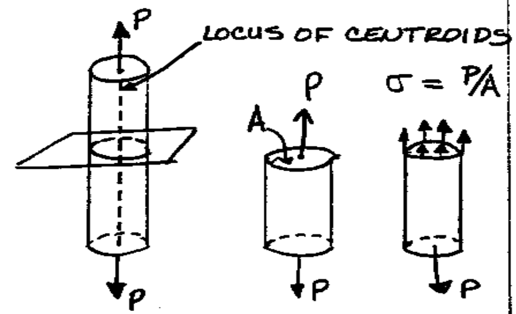


METHOD OF SECTIONS

THE STRESSES AND DEFORMATIONS DUE TO THESE INTERNAL FORCE COMPONENTS WILL NOW BE STUDIED ONE AT A TIME.

I. Axial Load — Normal Stress and Strain

Normal Stress



For an axially loaded prismatic bar, the normal stress on a plane perpindicular to the axis of the bar is given by:

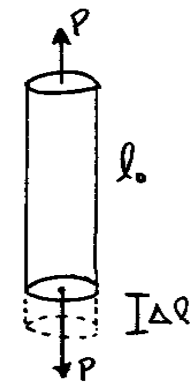
$$\sigma = P/A \qquad (1)$$

where σ = uniform normal stress

P = axial load (through the centroid)

A = cross sectional area

Extensional Strain



The extensional strain is

$$\epsilon = \Delta l / l_o \qquad (2)$$

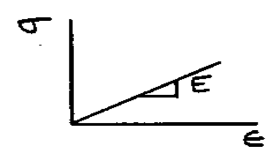
where ϵ = extensional strain

l_o = original length

Δl = change in length

Uniaxial Hooke's Law

For an isotropic, elastic material there is a linear relationship between stress and strain for uniaxial loads, $\therefore$ $\sigma = E\epsilon \quad (3)$



where E = Modulus of Elasticity or Young's Modulus

Deformation:

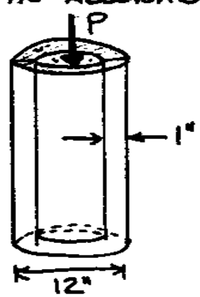
Substitution of (1) and (2) into (3) yields

$$P/A = E\frac{\Delta l}{l} \quad \text{or} \quad \Delta l = \frac{Pl}{AE}$$

If P, A, or E vary along the length then $\Delta l = \int \frac{P\,dx}{AE}$

AXIAL FORCE EXAMPLE

A STEEL CYLINDER 12 IN. OUTSIDE DIAMETER AND A WALL THICKNESS OF 1 IN. IS FILLED WITH CONCRETE AND USED AS A PIER TO SUPPORT AN AXIAL LOAD IN COMPRESSION. IF THE ALLOWABLE STRESSES ARE 40,000 PSI FOR STEEL AND 3500 PSI FOR CONCRETE AND THE YOUNG'S MODULUS FOR THE STEEL IS 30×10^6 PSI AND FOR THE CONCRETE IS 2.8×10^6 PSI, WHAT IS THE ALLOWABLE LOAD ON THE PIER?



TOTAL LOAD $P = P_{STEEL} + P_{CONCRETE}$

STEEL AND CONCRETE MUST DEFORM THE SAME AMOUNT UNDER P

$$\therefore \Delta l_s = \Delta l_c$$

$$\frac{P_s L}{A_s E_s} = \frac{P_c L}{A_c E_c}$$

$$\frac{\sigma_s L}{E_s} = \frac{\sigma_c L}{E_c}$$

$$\frac{\sigma_s L}{30 \times 10^6} = \frac{\sigma_c L}{2.8 \times 10^6} \Rightarrow \sigma_s = \frac{30}{2.8}\sigma_c$$

$\therefore$ FOR $\sigma_c = 3500$ THEN $\sigma_s = \frac{30}{2.8}(3500) = \underline{37,500 \text{ PSI}}$

AND THE STEEL IS NOT STRESSED TO ITS MAXIMUM

$$P = \sigma_c A_c + \sigma_s A_s$$

$$= \frac{30}{2.8}(3500)\frac{\pi}{4}(12^2 - 10^2) + (3500)\frac{\pi}{4}(10)^2$$

$$= (3500)\frac{\pi}{4}\left[\frac{30}{2.8}(44) + 100\right]$$

$$\underline{P = 1,571 \text{ KIPS}}$$

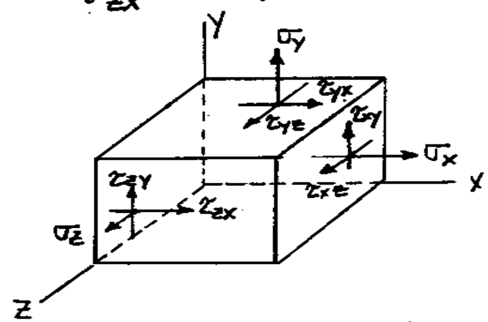
II. LINEAR STRESS-STRAIN LAW

GENERALIZED HOOKE'S LAW FOR ISOTROPIC, ELASTIC MATERIAL:

$$\epsilon_x = \frac{\sigma_x}{E} - \nu\frac{\sigma_y}{E} - \nu\frac{\sigma_z}{E} \qquad \gamma_{xy} = \tau_{xy}/G$$

$$\epsilon_y = \frac{\sigma_y}{E} - \nu\frac{\sigma_x}{E} - \nu\frac{\sigma_z}{E} \qquad \gamma_{yz} = \tau_{yz}/G$$

$$\epsilon_z = \frac{\sigma_z}{E} - \nu\frac{\sigma_x}{E} - \nu\frac{\sigma_y}{E} \qquad \gamma_{zx} = \tau_{zx}/G$$



WHERE ν = POISSON'S RATIO
G = SHEAR MODULUS
τ_{ij} = SHEAR STRESS ON i FACE IN j DIRECTION
σ_i = NORMAL STRESS ON i FACE
ϵ_i = EXTENSIONAL STRAIN IN i DIRECTION
γ_{ij} = SHEAR STRAIN BETWEEN i & j DIRECTIONS

E, G & ν ARE RELATED BY

$$G = \frac{E}{2(1+\nu)}$$

THUS, THERE ARE ONLY TWO INDEPENDENT MATERIAL PROPERTIES.

THERMAL STRAINS

FOR HOMOGENEOUS ISOTROPIC MATERIALS, A CHANGE IN TEMPERATURE OF ΔT DEGREES CAUSES UNIFORM LINEAR STRAINS IN EVERY DIRECTION GIVEN BY:

$$\epsilon_x = \epsilon_y = \epsilon_z = \alpha\Delta T$$

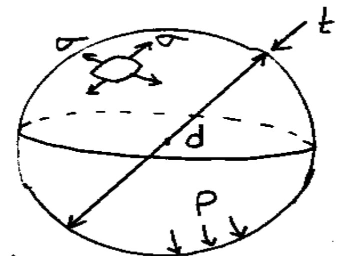
WHERE α = COEFFICIENT OF THERMAL EXPANSION FOR THE MATERIAL

THESE STRAINS MUST BE SUPERPOSED ON THE STRAINS DEFINED BY HOOKE'S LAW ABOVE.

II-A THIN-WALLED PRESSURE VESSELS

SPHERICAL PRESSURE VESSELS (INTERNAL PRESSURE)

AN ELEMENT IN THE WALL OF THE SPHERE WILL BE SUBJECTED TO EQUAL STRESSES FOR ALL ORIENTATIONS. THE STRESS IS GIVEN BY :



$$\sigma_\theta \equiv \sigma = \frac{Pr}{2t} = \frac{Pd}{4t}$$

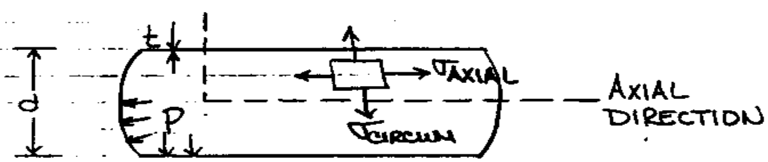
WHERE :
- P = INTERNAL PRESSURE (PSI)
- r = RADIUS OF SPHERE (IN)
- t = WALL THICKNESS (IN)

CYLINDRICAL PRESSURE VESSELS

AN ELEMENT SUFFICIENTLY REMOVED FROM THE HEADS WILL BE SUBJECTED TO THE FOLLOWING STRESSES :

$$\sigma_{AXIAL} = \frac{Pr}{2t} = \frac{Pd}{4t} \quad \text{(AXIAL DIRECTION)}$$

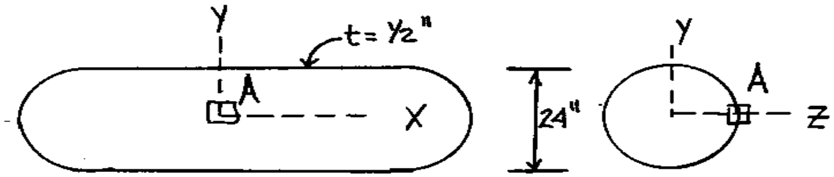
$$\sigma_{CIRCUM} = \frac{Pr}{t} = \frac{Pd}{2t} \quad \text{(CIRCUMFERENTIAL DIRECTION)}$$



CYLINDRICAL PRESSURE VESSEL & HOOKE'S LAW PROBLEM

— EIT EXAM, SPRING 1972

A STEEL PRESSURE VESSEL IS 24" O.D. WITH A 1/2" WALL THICKNESS. WHEN PRESSURIZED, THE STRAINS IN THE X AND Y DIRECTIONS AT POINT A WERE FOUND TO BE 0.0073×10^{-3} IN/IN AND 0.310×10^{-3} IN/IN, RESPECTIVELY. COMPUTE THE STRESS IN THE X AND Y DIRECTIONS AT POINT A AND THE MAGNITUDE OF THE INTERNAL PRESSURE. $E = 30 \times 10^6$ PSI, $\nu = 0.3$



$$\epsilon_x = 0.073 \times 10^{-3} \text{ IN/IN}, \quad \epsilon_y = 0.310 \times 10^{-3} \text{ IN/IN}$$

BIAXIAL HOOKE'S LAW

$$\epsilon_x = \frac{1}{E}\left[\sigma_x - \nu\sigma_y\right], \qquad \epsilon_y = \frac{1}{E}\left[\sigma_y - \nu\sigma_x\right]$$

SOLVE SIMULTANEOUSLY FOR σ_x & σ_y

$$\sigma_x = \frac{E}{1-\nu^2}\left(\epsilon_x + \nu\epsilon_y\right), \quad \sigma_y = \frac{E}{1-\nu^2}\left(\epsilon_y + \nu\epsilon_x\right)$$

$$\sigma_x = \frac{30 \times 10^6}{1-(0.3)^2}\left[73 + 0.3(310)\right]10^{-6} = \frac{30}{0.91}(73+93) = \underline{\underline{5,472 \text{ PSI}}}$$

$$\sigma_y = \frac{30 \times 10^6}{1-(0.3)^2}\left[310 + 0.3(73)\right]10^{-6} = \frac{30}{0.91}(310+21.9) = \underline{\underline{10,942 \text{ PSI}}}$$

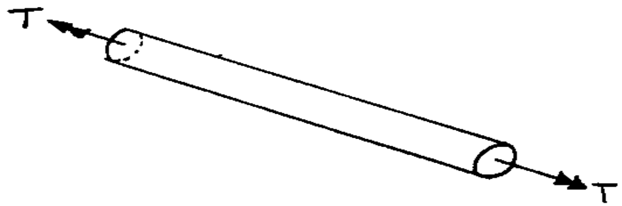
INTERNAL PRESSURE

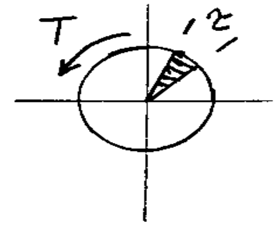
CIRCUMFERENTIAL STRESS $(\sigma_y) = \frac{Pr}{t}$

P = INTERNAL PRESSURE
r = AVERAGE RADIUS
t = WALL THICKNESS

$$P = \frac{\sigma_y t}{r} = \frac{10,942(1/2)}{12} = \underline{\underline{456 \text{ PSI}}}$$

III TORSION OF MEMBERS WITH CIRCULAR CROSS SECTIONS

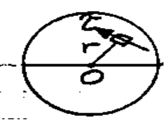




TORQUE (T) ON A CROSS SECTION CAUSES SHEAR STRESSES IN THE PLANE OF THE SECTION WHICH VARY LINEARLY WITH THE RADIUS AND ARE DIRECTED PERPINDICULAR TO THE RADIUS, I.E.

$$\tau = \frac{Tr}{J}$$

WHERE r IS THE RADIUS TO THE POINT IN QUESTION AND J IS THE POLAR MOMENT OF INERTIA RELATIVE TO THE CENTROID



$$J = \int_A r^2 dA = \frac{\pi r^4}{2} = \frac{\pi d^4}{32}$$

MAXIMUM TORSIONAL SHEAR STRESS OCCURS AT THE OUTER FIBERS (MAX r)

DEFORMATION

THE RELATIVE ANGLE OF TWIST BETWEEN TWO SECTIONS A DISTANCE ℓ APART IS GIVEN BY

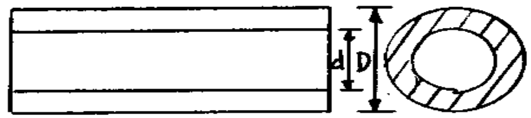
$$\phi = \frac{T\ell}{GJ}$$ OR (IF T, G, OR J VARY OVER ℓ)

$$\phi = \int_\ell \frac{T\,dx}{GJ}$$

WHERE G = SHEAR MODULUS

TORSION EXAMPLE

THE HOLLOW STEEL SHAFT ($G = 12 \times 10^6$ PSI) MUST TRANSMIT A TORQUE OF 300,000 IN-LB. THE TOTAL ANGLE OF TWIST MUST NOT EXCEED 3° AND τ_{MAX} MUST NOT EXCEED 16,000 PSI. FIND THE INSIDE DIAMETER (d) AND OUTSIDE DIAMETER (D) OF THE SHAFT THAT MEETS THESE CONDITIONS.



100"

STRESS

$$\tau = \frac{T\rho}{J}$$

$$16{,}000 = \frac{300{,}000\,(D/2)}{\frac{\pi}{32}(D^4 - d^4)}$$

TWIST

$$\phi = \frac{TL}{GJ}$$

$$3\left(\frac{\pi}{180}\right) = \frac{300{,}000\,(100)}{(12 \times 10^6)\,\pi/32\,(D^4 - d^4)}$$

$$\frac{T}{J} = \underbrace{\frac{\tau}{\rho} = \frac{\phi G}{L}}$$

$$\frac{16{,}000}{D/2} = \frac{3(\pi/180)(12 \times 10^6)}{100}$$

$$D = \frac{(16{,}000)(100)(2)}{3(\pi/180)(12 \times 10^6)}$$

$$D = 16/\pi = \underline{\underline{5.1''}}$$

$$J = \frac{T\rho}{\tau}$$

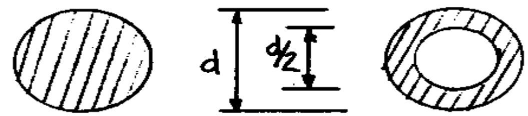
$$\pi/32\,(D^4 - d^4) = \frac{(300{,}000)\,\frac{16}{2\pi}}{16{,}000}$$

$$\pi/32\left(\left(\frac{16}{\pi}\right)^4 - d^4\right) = \frac{300{,}000\,(16/2\pi)}{16{,}000}$$

$$\underline{\underline{d = 3.72''}}$$

TORSION EXAMPLE

TWO CIRCULAR SHAFTS, ONE HOLLOW AND ONE SOLID, ARE MADE OF THE SAME MATERIAL AND HAVE THE DIAMETERS SHOWN. IF T_H IS THE TWISTING MOMENT THAT THE HOLLOW SHAFT CAN RESIST AND T_S IS THE TWISTING MOMENT THAT THE SOLID SHAFT CAN RESIST, DETERMINE THE RATIO OF T_H TO T_S.



SOLID

$$\tau_S = \frac{T_S \rho}{J_S} \qquad J_S = \frac{\pi d^4}{32}$$

$$\tau_S = \frac{T_S (d/2)}{\pi (d^4/32)} = \frac{16 T_S}{\pi d^3}$$

HOLLOW

$$\tau_H = \frac{T_H \rho}{J_H} \qquad J_H = \frac{\pi d^4}{32} - \frac{\pi (d/2)^4}{32} = \frac{\pi d^4}{32}\left(1 - 1/16\right)$$

$$= \frac{\pi d^4}{32}\left(\frac{15}{16}\right)$$

$$\tau_H = \frac{T_H (d/2)}{\frac{\pi d^4}{32}\left(\frac{15}{16}\right)} = \frac{16 T_H}{\pi d^3 (15/16)}$$

SINCE SAME MATERIAL $\tau_S = \tau_H$

$$\therefore \quad \frac{16 T_S}{\pi d^3} = \frac{16 T_H}{\pi d^3 (15/16)}$$

$$\boxed{\frac{T_H}{T_S} = \frac{15}{16}}$$

EIT REVIEW PROBLEMS

TORSION

1. A CYLINDRICAL SHAFT 25 FT. LONG HAS A DIAMETER OF 3 IN. THROUGH WHAT ANGLE MAY ONE END BE TWISTED WITH RESPECT TO THE OTHER WITHOUT EXCEEDING A SHEARING STRESS OF 10,000 PSI ? $G = 12 \times 10^6$ PSI

SHEAR STRESS : $\tau = T\rho / J$

ANGLE OF TWIST : $\theta = \int_L \frac{T\,dx}{GJ} = \frac{TL}{GJ} \Longrightarrow \frac{T}{J} = \frac{\theta G}{L}$

$$\tau = \left(\frac{\theta G}{L}\right)\rho \Rightarrow 10{,}000 = \frac{\theta\,(12 \times 10^6)(3/2)}{}$$

$$\boxed{\theta = 0.1667 \text{ RAD.}}$$

2. A SOLID CIRCULAR STEEL SHAFT 18 FT. LONG TRANSMITS 94,248 HP @ 180 RPM. THE ALLOWABLE SHEARING STRESS IS 10,000 PSI AND THE MAXIMUM ALLOWABLE ANGLE OF TWIST IS 0.040 RADIANS. $G = 12 \times 10^6$. DETERMINE THE MINIMUM ALLOWABLE DIAMETER OF THE SHAFT.

$$T = \frac{63{,}000\,HP}{N} \text{ in}\cdot\text{lb} \quad (N = RPM)$$

$$= \frac{63{,}000(94{,}248)}{180} = 33{,}000 \text{ in}\cdot\text{lb}$$

d AS GOVERNED BY τ :

$$\tau = \frac{T\rho}{J} \Rightarrow 10{,}000 = \frac{(33{,}000)(d/2)}{\pi d^4/32}$$

$$d^3 = \frac{33{,}000(16)}{10{,}000\,\pi} = 17 \quad \therefore \underline{\underline{d = 2.6 \text{ IN}}}$$

d AS GOVERNED BY θ :

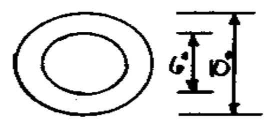
$$\theta = \frac{TL}{GJ} \Longrightarrow 0.040 = \frac{33{,}000\,(120)}{(12 \times 10^6)\pi d^4/32}$$

$$d^4 = \frac{33{,}000(120)(32)}{(12 \times 10^6)(4 \times 10^{-2})\pi} = 84 \quad \therefore \boxed{d = 4.4 \text{ IN}}$$

USE

TORSION PROBLEM FROM EIT EXAM - SPRING, 1972

A HOLLOW STEEL SHAFT HAS AN EXTERNAL DIAMETER OF 10" AND AN INTERNAL DIAMETER OF 6". DETERMINE THE HORSEPOWER THE SHAFT CAN TRANSMIT WHILE ROTATING AT 90 R.P.M. IF THE MAX. ALLOWABLE SHEARING STRESS IS 8000 PSI AND THE ALLOWABLE ANGLE OF TWIST IS 1.5° IN 15 FT. THE SHEAR MODULUS OF ELASTICITY IS 12×10^6 PSI.



SOLUTION

1. FIND THE ALLOWABLE TORQUE. ASSUME τ_{MAX} GOVERNS AND THEN CHECK TO SEE IF ϕ_{ALLOW} IS EXCEEDED.

$$\tau_{MAX} = \frac{Tr}{J}$$

$\tau_{MAX} = 8000$ PSI

$T = ?$

$r = r_{MAX} = 5"$

$J = \frac{\pi}{32}(d_o^4 - d_i^4) = \frac{\pi}{32}(10^4 - 6^4)$

$= 854.5 \text{ in}^4$

$$8000 = \frac{T(5)}{854.5}$$

$$\underline{T = 1{,}367{,}000 \text{ in-lb}}$$

CHECK ϕ :

$$\phi = \frac{TL}{GJ} \longrightarrow \phi = \frac{\left[\frac{(8000)(854.5)}{5}\right](15)(12)}{(12\times10^6)(854.5)} \text{ RAD} \left(\frac{180}{\pi}\right)$$

$$= 1.375° < 1.5° \quad \underline{OK}$$

2. FIND ALLOWABLE HP.

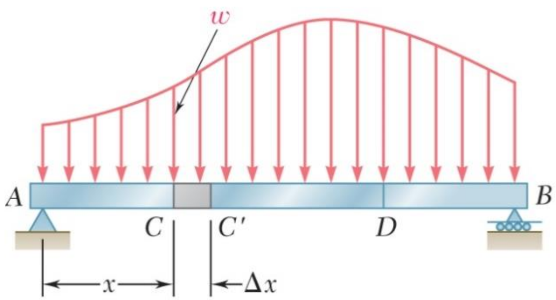
$$T = \frac{63{,}000 \text{ HP}}{n}$$

T = TORQUE IN IN-LB
HP = HORSEPOWER
n = SPEED IN RPM

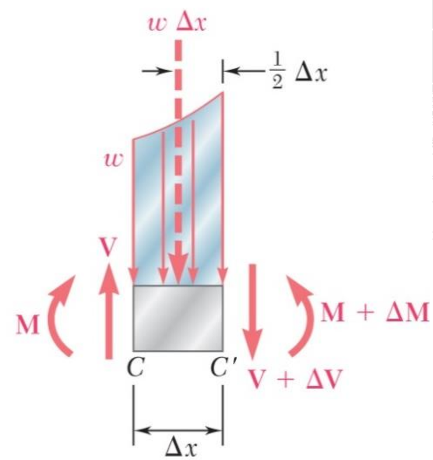
$$HP = \frac{Tn}{63{,}000} = \frac{(1{,}367{,}000)(90)}{63{,}000} = \underline{1953 \text{ HP}}$$

IV. SHEAR AND BENDING MOMENT IN BEAMS

AT ANY SECTION IN A BEAM LOADED BY TRANSVERSE LOADS THERE WILL, IN GENERAL, EXIST A SHEAR FORCE AND BENDING MOMENT. A FREE BODY DIAGRAM OF THE PORTION OF THE BEAM TO ONE SIDE OF THE SECTION WILL PERMIT THE DETERMINATION OF THIS INTERNAL FORCE SYSTEM. POSITIVE SIGN CONVENTIONS ARE SHOWN ON THE FIGURE BELOW.



(a)



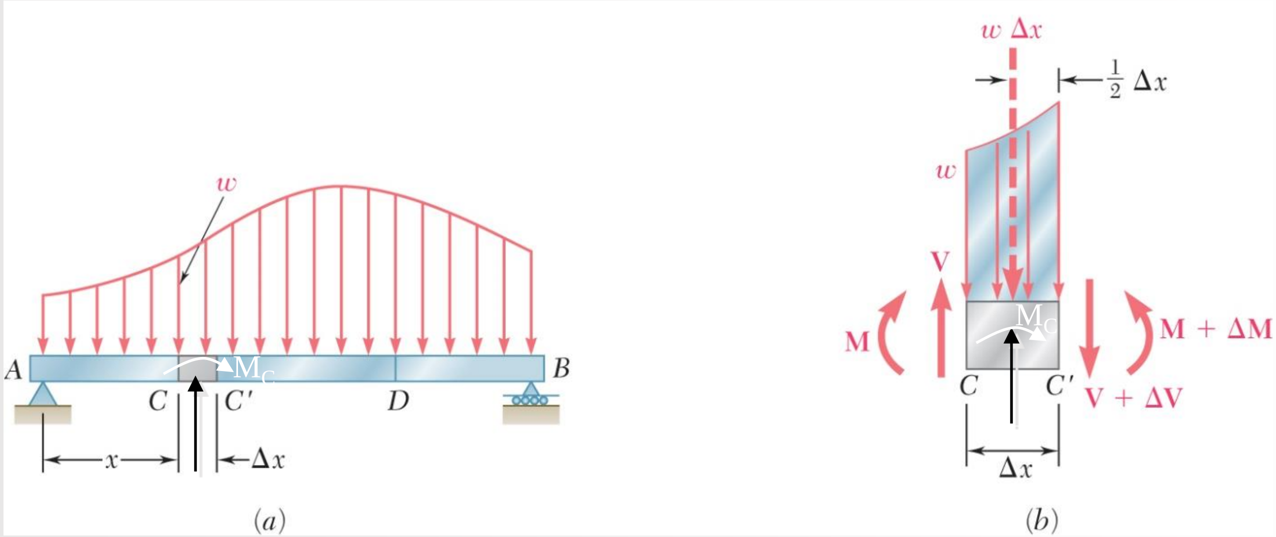
(b)

$dV/dx = \boxed{-w}$ → SLOPE OF V DIAGRAM EQUALS THE NEGATIVE OF THE ORDINATE OF THE LOAD DIAGRAM.

$dV = \boxed{-w\,dx}$ → CHANGE IN V EQUALS THE NEGATIVE OF THE AREA UNDER THE LOAD DIAGRAM.

$dM/dx = \boxed{V}$ → *SLOPE OF M DIAGRAM EQUALS THE POSITIVE OF THE ORDINATE OF THE SHEAR DIAGRAM*

$dM = \boxed{Vdx}$ → *CHANGE IN M EQUALS THE POSITIVE OF THE AREA UNDER THE SHEAR DIAGRAM*



IF THERE ARE CONCENTERATED FORCES/MOMENTS ACTING AT A POINT,

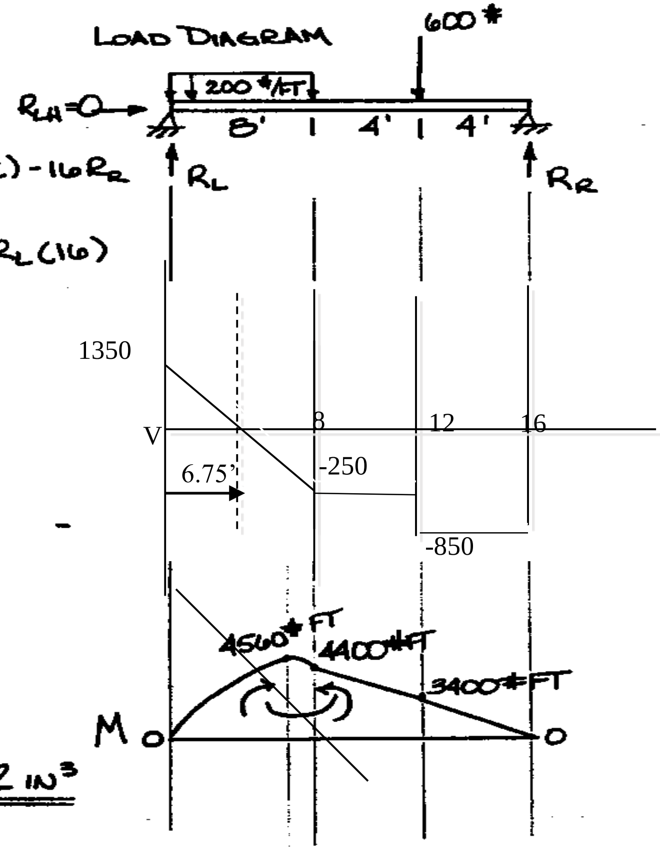
$$V + \Delta V = V + F_C(\uparrow)$$
$$M + \Delta M = M + M_C(\curvearrowright)$$

SHEAR & MOMENT IN BEAMS

A 16 FT. BEAM HAS SIMPLE SUPPORTS AT THE ENDS AND CARRIES A LOAD AS INDICATED IN THE FIGURE.

(a) SKETCH NEATLY TO AN APPROXIMATE SCALE THE LOADING, SHEAR, AND BENDING MOMENT DIAGRAMS. WRITE THE MAGNITUDES OF THE ESSENTIAL VALUES ADJACENT TO WHERE THEY OCCUR ON EACH DIAGRAM. GIVE THE DISTANCE AS A DIMENSION FROM THE LEFT END OF THE BEAM TO WHERE THESE VALUES OCCUR.

(b) EVALUATE THE SECTION MODULUS FOR AN ALLOWABLE BENDING OF 30,000 PSI.



SOLVE FOR REACTIONS

$\Sigma M_L = 0 = 200(8)(8/2) + 600(12) - 16R_R$

$\underline{R_R = 850\#\uparrow}$

$\Sigma M_R = 0 = 600(4) + 200(8)(8/2) - R_L(16)$

$\underline{R_L = 1350\#\uparrow}$

(b) $\sigma = M/S$

$S = M/\sigma$

M = MAXIMUM MOMENT

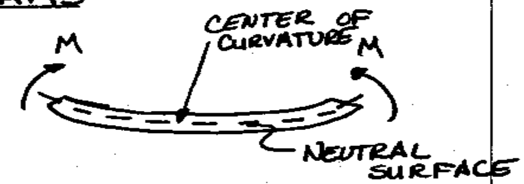
$S = I/c$ = SECTION MODULUS

σ = ALLOWABLE STRESS

$$S = \frac{(4560\ \#FT)(12\ IN/FT)}{30,000} = \underline{\underline{1.82\ IN^3}}$$

V. FLEXURAL STRESSES IN BEAMS

BENDING MOMENTS, M, ON A BEAM SEGMENT CAUSE DEFORMATIONS AS SHOWN AND THE RADIUS OF CURVATURE, ρ, AND M ARE RELATED BY:

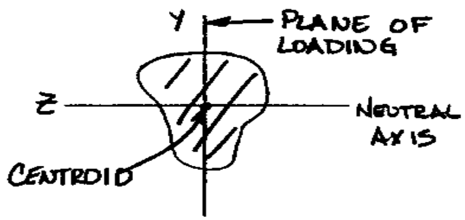


$$1/\rho = \frac{M}{EI}$$ WHERE

E = MODULUS OF ELASTICITY

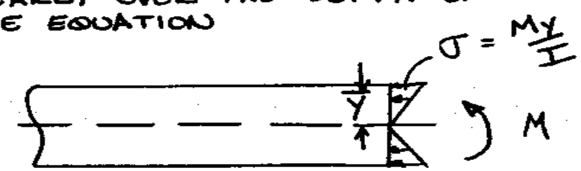
I = MOMENT OF INERTIA ABOUT THE NEUTRAL AXIS OF THE CROSS SECTION

$$I = \int_A y^2 dA$$



FLEXURAL STRESSES VARY LINEARLY OVER THE DEPTH OF THE BEAM ACCORDING TO THE EQUATION

$$\sigma = -\frac{My}{I}$$



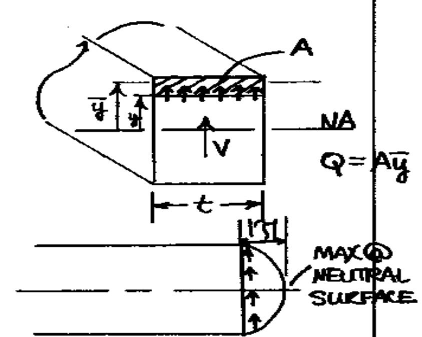
WHERE Y IS THE DISTANCE FROM THE NEUTRAL AXIS (+ UPWARDS)

FOR THE FLEXURAL FORMULA TO BE VALID THE PLANE OF LOADING MUST BE A PRINCIPAL PLANE.

VI. SHEARING STRESSES IN BEAMS

BEAM SHEAR STRESS IS GIVEN BY:

$$\tau = \frac{VQ}{It}$$

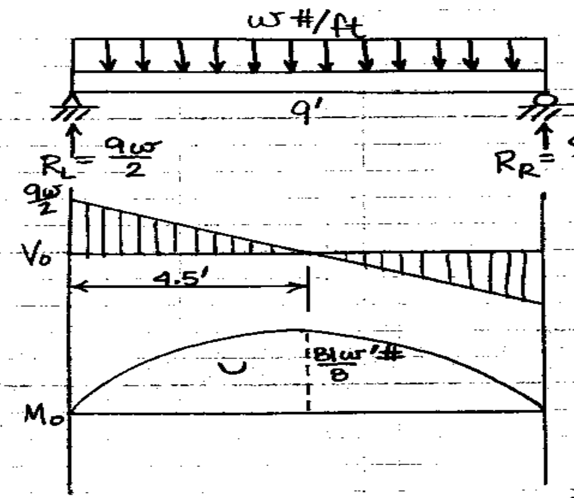


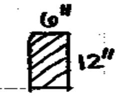
WHERE t = WIDTH OF SECTION WHERE τ IS DETERMINED

Q = FIRST MOMENT OF THE AREA ISOLATED WITH THE FIRST MOMENT TAKEN ABOUT THE NEUTRAL AXIS

BEAM STRESSES — EIT EXAM, SPRING, 1972

THE SIMPLY SUPPORTED BEAM IS 6" WIDE, 12" DEEP AND CARRIES A UNIFORMLY DISTRIBUTED LOAD OVER ITS ENTIRE 9 FT. LENGTH. THE ALLOWABLE BENDING STRESS FOR THE BEAM IS 1800 PSI IN TENSION OR COMPRESSION AND THE ALLOWABLE HORIZONTAL SHEAR STRESS IS 120 PSI. DETERMINE THE MAX. ALLOWABLE UNIFORM LOAD, ω, ON THE BEAM.





$$V_{max} = \frac{9\omega}{2} \ \#$$

$$M_{max} = \frac{81\omega}{8} \ '\#$$

a) BENDING

$$I = \frac{bh^3}{12} = \frac{6(12)^3}{12} = \underline{864 \text{ in}^4}$$

$$\sigma_{max} = \frac{Mc}{I}$$

$$1800 = \frac{\left(\frac{81\omega}{8}\right)(12)(6)}{864} \ ; \quad \underline{\omega = 2133 \ \#/\text{ft}}$$

b) SHEAR

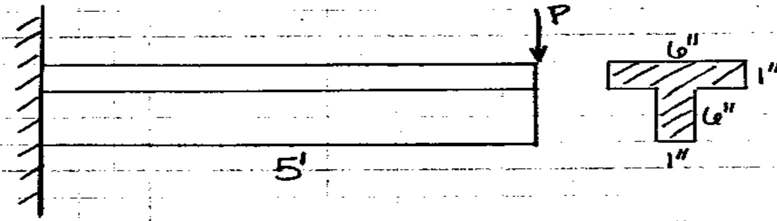
$$\tau_{max} = \frac{VQ_{max}}{Ib} \qquad Q_{max} = A\bar{y} \ @ \text{N.A.}$$

$$120 = \frac{\left(\frac{9\omega}{2}\right)\left[\overbrace{(6)(6)}^{A}\overbrace{(3)}^{\bar{y}}\right]}{864(6)} \ ; \quad \underline{\omega = 1280 \ \#/\text{ft}}$$

$\therefore$ SHEAR GOVERNS AND $\underline{\underline{\omega_{ALLOW} = 1280 \ \#/\text{ft}}}$

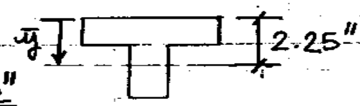
FLEXURE EXAMPLE

GIVEN THE BEAM SHOWN BELOW. THE ALLOWABLE TENSILE UNIT STRESS IS 3000 PSI ; THE ALLOWABLE COMPRESSIVE UNIT STRESS IS 16,000 PSI. NEGLECTING THE WEIGHT OF THE BEAM, WHAT IS THE MAXIMUM SAFE CONCENTRATED LOAD THAT MAY BE PLACED AT THE FREE END OF THE BEAM?



SOLUTION:

LOCATE NEUTRAL AXIS

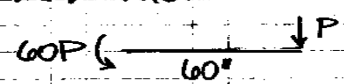


$$\bar{y} = \frac{\Sigma \bar{y}_i A_i}{\Sigma A_i} = \frac{6(1/2) + 6(4)}{12} = \underline{\underline{2.25''}}$$

DETERMINE I_{NA}

$$I = \Sigma\left[I_c + Ad^2\right] = \frac{6(1)^3}{12} + 6(1.75)^2 + \frac{(1)(6)^3}{12} + 6(1.75)^2 = \underline{\underline{55.25 \text{ in}^4}}$$

MAX. MOMENT = $-5P'^{\#} = -60P''^{\#}$ @ FIXED END



TENSION @ TOP FIBERS

$$\sigma = \frac{My}{I}\ ;\ 3000 = \frac{60P(2.25)}{55.25}$$

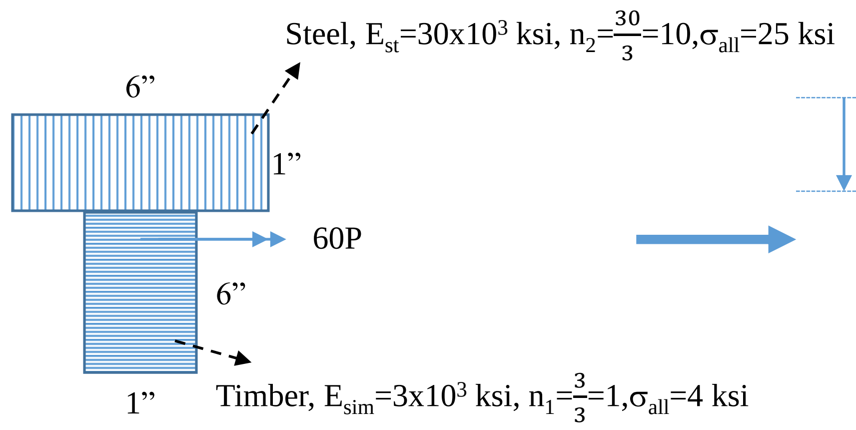
$\underline{\underline{P = 1227^{\#}}}$ ← TENSION GOVERNS

COMPRESSION @ BOTTOM

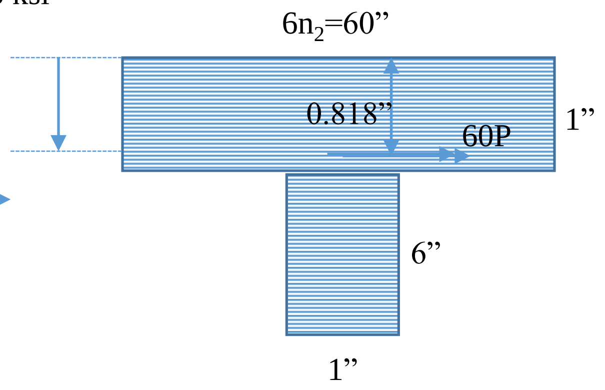
$$\sigma = \frac{My}{I} \Rightarrow 1600 = \frac{60P(4.75)}{55.25}$$

$\underline{\underline{P = 3102^{\#}}}$

For a composite X-section



Original X-section



Transformed X-section

$$\bar{y} = \frac{60 \times \left(\frac{1}{2}\right) + 6 \times 4}{60 + 6} = 0.818, \bar{I} = \frac{60 \times 1^3}{12} + 60 \times (0.818 - 0.5)^2 + \frac{1 \times 6^3}{12} + 6 \times (4 - 0.818)^2 = 89.82\ in^4$$

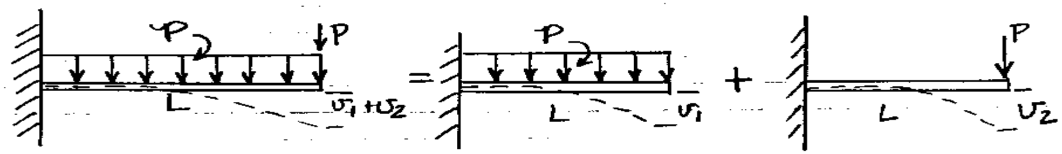
$$n_1 \frac{60P \times (6 - 0.818)}{89.82} = 4 \Rightarrow P = 1.15\ kip, n_2 \frac{60P \times (0.818)}{89.82} = 25 \Rightarrow P = 4.57\ kip$$

$$P = 1.15\ kip\ (timber\ governs)$$

VII. DEFLECTION OF BEAMS BY SUPERPOSITION

THE METHOD OF OBTAINING A RESULTANT EFFECT BY ADDING TOGETHER PARTIAL EFFECTS IS CALLED THE METHOD OF SUPERPOSITION. IT IS WIDELY USED IN INVESTIGATING BENDING OF BEAMS AND, IN THIS CONNECTION, IT CONSISTS OF UTILIZING THE RESULTS OF A FEW SIMPLE DEFLECTION PROBLEMS OBTAINED PREVIOUSLY TO BUILD UP THE SOLUTIONS OF MORE COMPLICATED PROBLEMS.

CONSIDER THE CANTILEVER BEAM BELOW. THE LOADING OF THE BEAM CAN BE REPRESENTED BY ELEMENTAL PARTS WHICH CORRESPOND TO CASES SHOWN ON THE FOLLOWING PAGES.

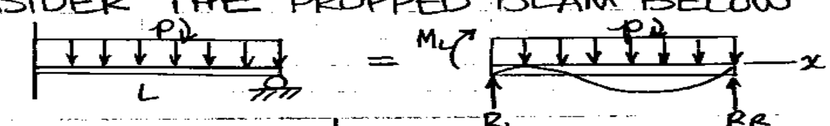


THUS, THE RESULTANT DEFORMATIONS CAN BE DETERMINED BY COMBINATION OF THE DEFORMATIONS DUE TO THE INDIVIDUAL LOADS.

VIII. STATICALLY INDETERMINATE BEAMS BY SUPERPOSITION

THE METHOD OF SUPERPOSITION CAN BE USED FOR STATICALLY INDETERMINATE PROBLEMS IN BENDING. THE SOLUTIONS WILL BE OBTAINED BY COMBINING STATICALLY DETERMINATE CASES IN SUCH A MANNER AS TO SATISFY THE CONDITIONS OF THE SUPPORTS:

CONSIDER THE PROPPED BEAM BELOW



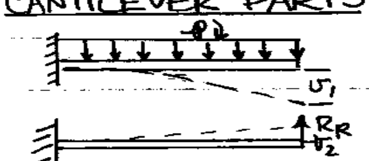
SUPPORT CONDITIONS

$v(0)=0$

$v(L)=0$

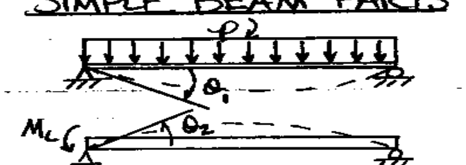
$\theta(0)=0$

CANTILEVER PARTS



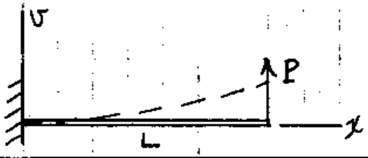
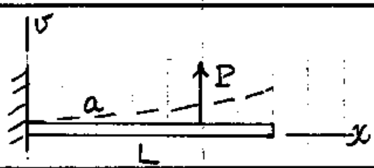
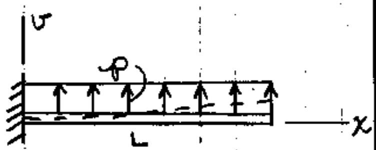
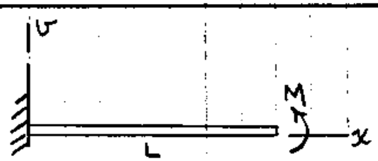
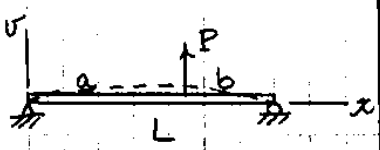
$v_1+v_2=0$

SIMPLE BEAM PARTS

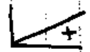


$\theta_1+\theta_2=0$

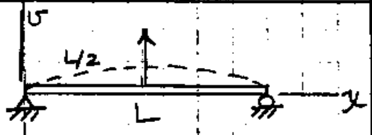
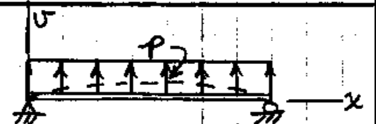
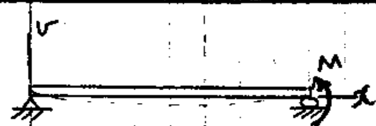
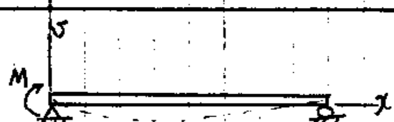
SUMMARY OF BEAM LOADINGS

CASE NO.	TYPE OF LOAD	SLOPE EQUATION	DEFLECTION EQUATION	MAX. DEFLECTION
1		$\theta(x)=\frac{P}{2EI}\left[2Lx-x^2\right]$	$v(x)=\frac{P}{6EI}\left[3Lx^2-x^3\right]$	$v(L)=\frac{PL^3}{3EI}$
2		$\underline{0<x<a}$ $\theta(x)=\frac{P}{2EI}\left[2ax-x^2\right]$ $\underline{a<x<L}$ $\theta(x)=\frac{Pa^2}{2EI}$	$\underline{0<x<a}$ $v(x)=\frac{P}{6EI}\left[3ax^2-x^3\right]$ $\underline{a<x<L}$ $v(x)=\frac{Pa^2}{6EI}(3x-a)$	$v(L)=\frac{Pa^2}{6EI}(3L-a)$
3		$\theta(x)=\frac{\wp}{6EI}\left[3L^2x-3Lx^2+x^3\right]$	$v(x)=\frac{\wp}{24EI}\left[6L^2x^2-4Lx^3+x^4\right]$	$v(L)=\frac{\wp L^4}{8EI}$
4		$\theta(x)=\frac{Mx}{EI}$	$v(x)=\frac{Mx^2}{2EI}$	$v(L)=\frac{ML^2}{2EI}$
5		$\underline{0<x<a}$ $\theta(x)=\frac{Pb}{6EIL}\left[L^2-3x^2-b^2\right]$ $\underline{a<x<L}$ $\theta(x)=\frac{Pb}{6EIL}\left[\frac{3L}{b}(x-a)^2+(L^2-b^2)-3x^2\right]$	$\underline{0<x<a}$ $v(x)=\frac{Pb}{6EIL}\left[L^2x-x^3-b^2x\right]$ $\underline{a<x<L}$ $v(x)=\frac{Pb}{6EIL}\left[\frac{L}{b}(x-a)^3+(L^2-b^2)x-x^3\right]$	$\underline{a>b}$ $v\left(\frac{\sqrt{L^2-b^2}}{2}\right)=\frac{Pb}{9\sqrt{3}EIL}(L^2-b^2)^{3/2}$

SUMMARY OF BEAM LOADINGS



+υ = UPWARD

CASE NO.	TYPE OF LOADING	SLOPE EQUATION	DEFLECTION EQUATION	MAX. DEFLECTION
6		$0<x<L/2$ $\theta(x)=\frac{P}{4EI}\left[\frac{EL^2}{4}-x^2\right]$	$0<x<L/2$ $\upsilon(x)=\frac{P}{12EI}\left[\frac{3L^2x}{4}-x^3\right]$	$\upsilon(L/2)=\frac{PL^3}{48EI}$
7		$\theta(x)=\frac{\wp}{24EI}\left[L^3-6Lx^2+4x^3\right]$	$\upsilon(x)=\frac{x}{24EI}\left[L^3x-2Lx^3+x^4\right]$	$\upsilon(L/2)=\frac{5\wp L^4}{384EI}$
8			$0<x<a$ $\upsilon(x)=\frac{\wp}{24EIL}\left[4L^2ax^2-4La^3x+a^4x-4Lax^3+2a^2x^3+Lx^4\right]$ $a<x<L$ $\upsilon(x)=\frac{\wp a^2}{24EIL}\left[4L^2x-6Lx^2-La^2+a^2x+2x^3\right]$	
9		$\theta(x)=\frac{M}{6EIL}\left[3x^2-L^2\right]$	$\upsilon(x)=\frac{M}{6EIL}\left[x^3-L^2x\right]$	$\upsilon\left(\frac{L}{\sqrt{3}}\right)=\frac{-ML^2}{9\sqrt{3}EI}$
10		$\theta(x)=\frac{M}{6EIL}\left[6Lx-2L^2-3x^2\right]$	$\upsilon(x)=\frac{M}{6EIL}\left[3Lx^2-2L^2x-x^3\right]$	$\upsilon\left(L-\frac{L}{\sqrt{3}}\right)=\frac{-ML^2}{9\sqrt{3}EI}$

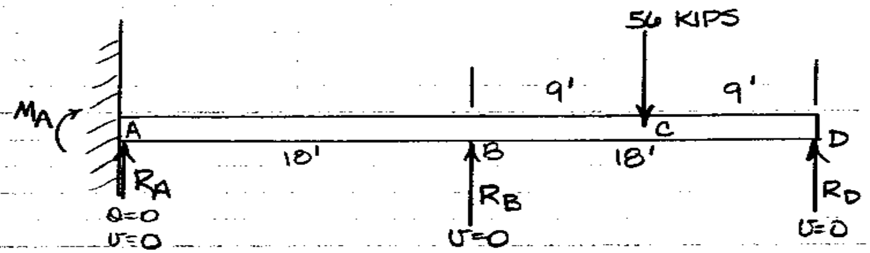
STATICALLY INDETERMINATE BEAM EXAMPLE

GIVEN THE CONTINUOUS BEAM A-B-D WITH UNIFORM MOMENT OF INERTIA FROM A TO D OF $1200\,in^4$ AND $E = 30 \times 10^6$ PSI.

DETERMINE: a) REACTION AT B

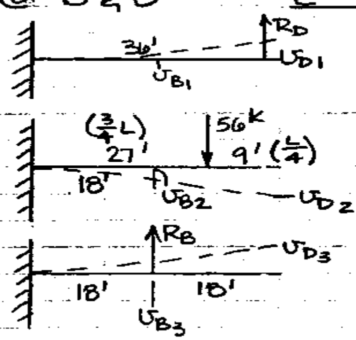
b) MOMENT CURVE, WITH ALL ORDINATES

c) DIRECTION AND SLOPE OF THE BEAM AT C IN RADIANS



THERE ARE FOUR UNKNOWN REACTIONS AND ONLY 2 INDEPENDANT EQUATIONS OF STATICS.

a) USE CANTILEVER PARTS AND SUPPORT CONDITIONS $v = 0$ @ B & D $\quad L = 36'$



FROM CASE 1

$$v_{D1} = \frac{R_D}{6EI}\left[3L^3 - L^3\right] = \frac{R_D L^3}{3EI}$$

$$v_{B1} = \frac{R_D}{6EI}\left[3L(L/2)^2 - (L/2)^3\right] = \frac{5 R_D L^3}{48EI}$$

FROM CASE 2

$$v_{D2} = \frac{P}{6EI}\left(\tfrac{3}{4}L\right)^2\left[3L - \tfrac{3}{4}L\right] = \frac{27(-56)L^3}{128EI} = \frac{-189L^3}{16EI}$$

$$v_{B2} = \frac{(-56)}{6EI}\left[3\left(\tfrac{3}{4}L\right)\left(\tfrac{L}{2}\right)^2 - \left(\tfrac{L}{2}\right)^3\right] = \frac{-49L^3}{12EI}$$

FROM CASE 3

$$v_{D3} = \frac{R_B}{6EI}(L/2)^2(3L - L/2) = \frac{5 R_B L^3}{48EI}$$

$$v_{B3} = \frac{R_B}{6EI}\left[3\left(\tfrac{L}{2}\right)\left(\tfrac{L}{2}\right)^2 - \left(\tfrac{L}{3}\right)^3\right] = \frac{R_B L^3}{24EI}$$

EQ. ①

$$v_D = v_{D1} + v_{D2} + v_{D3} = 0$$

$$\frac{R_D L^3}{3EI} - \frac{189L^3}{16EI} + \frac{5R_B L^3}{48EI} = 0$$

EQ. ②

$$v_B = v_{B1} + v_{B2} + v_{B3} = 0$$

$$\frac{5R_D L^3}{48EI} - \frac{49L^3}{12EI} + \frac{R_B L^3}{24EI} = 0$$

SOLUTION OF EQ'NS ① & ② YIELDS

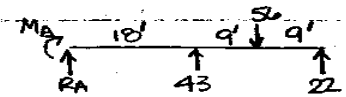
$\boxed{R_B = 43^k}$ ← ANSWER TO a)

$\boxed{R_D = 22^k}$

STATICALLY INDET. BEAM EXAMPLE CONT

b) MOMENT CURVE

NOW THAT R_B AND R_D ARE KNOWN, R_A AND M_A CAN BE DETERMINED BY STATICS.

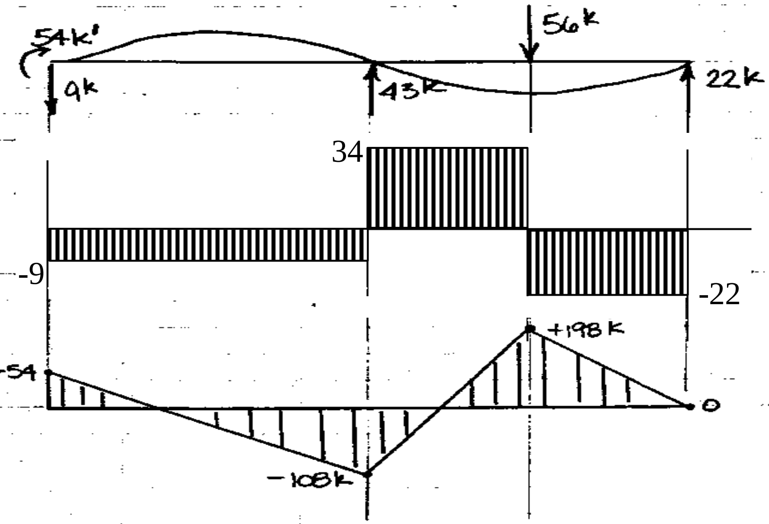


$$\Sigma F_V = 0 = R_A + 43 + 22 - 56$$
$$\therefore \underline{R_A = -9k}$$
$$\Sigma M_A = 0 = M_A + 27(56) - 18(43) - 36(22)$$
$$\therefore \underline{M_A = 54k'}$$

LOAD DIAG.

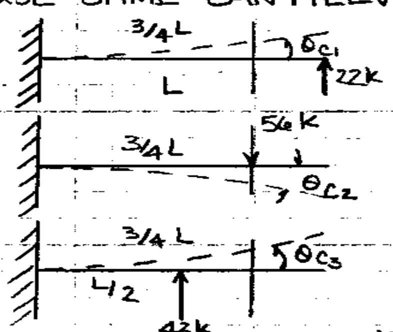
SHEAR DIAG. V

MOMENT DIAG. M



c) SLOPE AT C

USE SAME CANTILEVER PARTS AS IN a)



CASE 1

$$\theta_{C1} = \frac{(22)}{2EI}\left[2L\left(\tfrac{3}{4}L\right) - \left(\tfrac{3}{4}L\right)^2\right] = \frac{15(22)L^2}{32EI}$$

CASE 2

$$\theta_{C2} = \frac{(-56)}{2EI}\left(\tfrac{3}{4}L\right)^2 = \frac{-9L^2(56)}{32\,EI}$$

$$\theta_{C3} = \frac{(43)}{2EI}\left(\frac{L}{2}\right)^2 = \frac{43L^2}{8EI}$$

$$\theta_C = \theta_{C1} + \theta_{C2} + \theta_{C3} = \frac{15(22)L^2}{32EI} - \frac{9(56)L^2}{32EI} + \frac{43L^2}{8EI} = \frac{-L^2}{16EI}$$

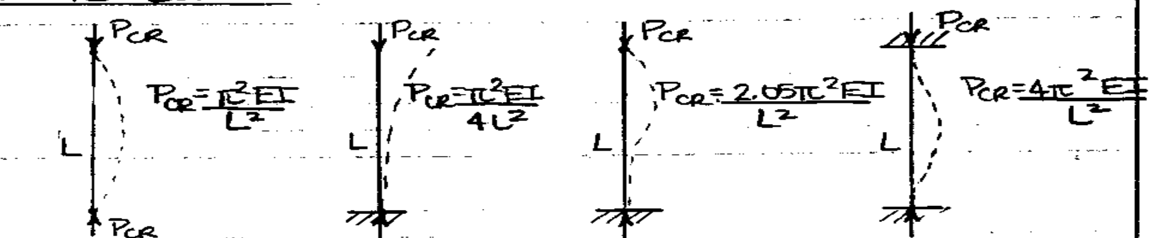
$L = 36'$, $E = 30\times10^3$ KSI, $I = 1200 in^4$

$$\theta_C = -\frac{(36)^2(144)}{16(30\times10^3)(1200)} = 325\times10^{-6} = \underline{0.000325 \text{ RAD}}$$

IX. EULER COLUMN EQUATION

WHEN THE LENGTH OF A COMPRESSION MEMBER IS LARGE IN COMPARISON WITH ITS TRANSVERSE DIMENSIONS, FAILURE TENDS TO OCCUR BY BUCKLING OR LATERAL BENDING RATHER THAN BY DIRECT COMPRESSION. EULER'S COLUMN EQUATION PREDICTS THE CRITICAL LOAD (P_{CR}) WHICH CAUSES BUCKLING AND DOES NOT DEPEND UPON THE STRENGTH OF THE MATERIAL BUT ONLY UPON THE DIMENSIONS OF THE STRUCTURE AND THE MODULUS OF ELASTICITY OF THE MATERIAL.

CRITICAL LOADS



PINNED ENDS — FLAG POLE — FIXED-PINNED — FIXED-FIXED

THE ABOVE FORMULAS ARE VALID ONLY FOR STRESSES UP TO THE PROPORTIONAL LIMIT OF THE MATERIAL.

CONSIDER THE PINNED-END CASE

$$\sigma_{CR} = \frac{P_{CR}}{A} = \frac{\pi^2 EI}{AL^2} = \frac{\pi^2 E}{L^2}\left(\frac{I}{A}\right) = \frac{\pi^2 E\left(\sqrt{I/A}\right)^2}{L^2} = \frac{\pi^2 E r^2}{L^2} = \frac{\pi^2 E}{(L/r)^2}$$

THIS EQUATION IS APPLICABLE AS LONG AS THE STRESS σ_{CR} REMAINS WITHIN THE PROPORTIONAL LIMIT.

FOR STEEL WITH $\sigma_{P.L.} = 30{,}000$ PSI
$E = 30 \times 10^6$ PSI

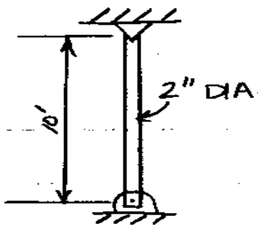
$$30{,}000 = \frac{\pi^2(30 \times 10^6)}{(L/r)^2} \Rightarrow L/r = \sqrt{1000\pi^2} \cong 100$$

$\therefore$ THE SLENDERNESS RATIO L/r MUST BE ≥ 100 FOR EULER'S EQUATION TO BE VALID.

EULER COLUMN EQUATION

A 2-INCH DIAMETER STEEL ROD, AB, 10 FT. LONG IS PINNED AT BOTH ENDS AND IS UNSTRESSED AT 60°F. DETERMINE THE HIGHEST TEMPERATURE TO WHICH THE BAR MAY BE HEATED BEFORE IT WILL BUCKLE. NEGLECT THE WEIGHT OF THE BAR.

$\alpha = 4 \times 10^{-6}/°F \quad , \quad E = 30 \times 10^{6}\ PSI$



CRITICAL STRESS (EULER)

$$\sigma_{CR} = \frac{P_{CR}}{A} = \frac{\pi^2 E}{(L/r)^2} = \frac{\pi^2 (30 \times 10^6)}{(10(12)/\frac{1}{\sqrt{2}})^2} = 82{,}247\ PSI$$

$$\frac{I}{A} = r^2$$

$$\left(\frac{\frac{\pi(2)^4}{32}}{\frac{\pi(2)^2}{4}}\right) = r^2$$

$$\left(\frac{(2)^2}{8}\right)^2 = r^2$$

$$\therefore r^2 = (1/2)$$

$$r = 1/\sqrt{2}$$

σ DUE TO HEATING:

$\sigma = E\epsilon$ (HOOKE'S LAW FOR AXIAL LOAD)

$\epsilon = \alpha \Delta T$ (THERMAL STRAIN)

$\therefore \sigma = E \alpha \Delta T$ EQUATE TO CRITICAL STRESS AND SOLVE FOR ΔT

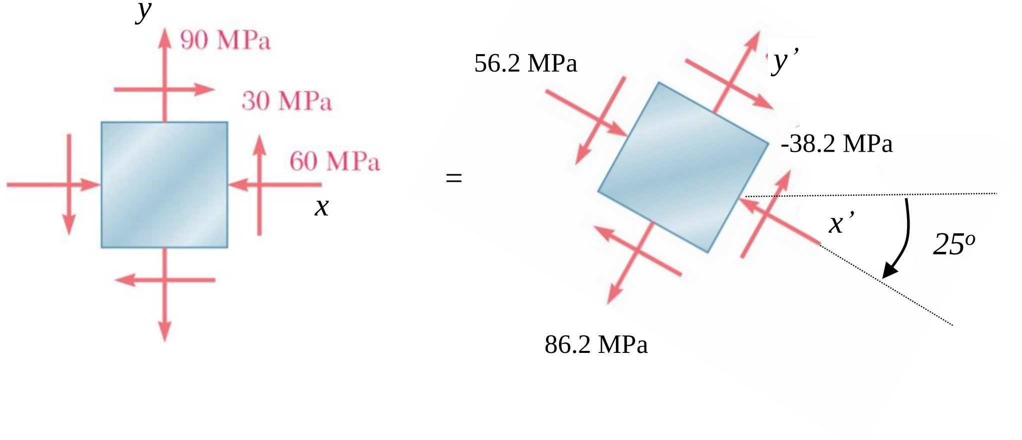
$$E \alpha \Delta T = \frac{\pi^2 E}{(L/r)^2}$$

$$\Delta T = \frac{\pi^2}{\alpha (L/r)^2} = \frac{\pi^2}{(4 \times 10^{-6})(120/\frac{1}{\sqrt{2}})^2} = 85.67°F$$

$T = 85.7 + 60 = \underline{145.7°F}$ MAX TEMP

STRESS TRANSFORMATION

$$\sigma = \left(\frac{\sigma_x + \sigma_y}{2}\right) + \left(\frac{\sigma_x - \sigma_y}{2}\right)\cos(2\theta) + \tau_{xy}\sin(2\theta), \tau = -\left(\frac{\sigma_x - \sigma_y}{2}\right)\sin(2\theta) \ +\tau_{xy}\cos(2\theta)$$



$$\sigma_x = -60\ (MPa), \sigma_y = 90\ (MPa), \tau_{xy} = 30\ (MPa)$$

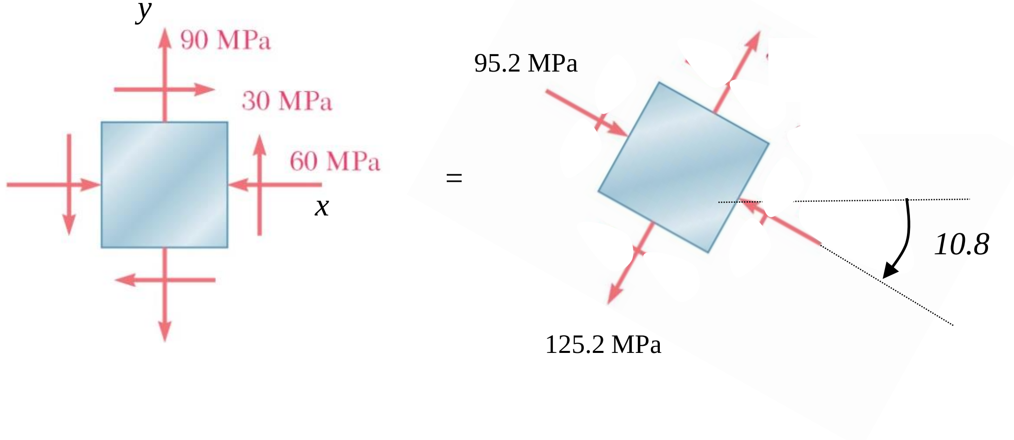
$$\sigma_{x'} = \left(\frac{-60 + 90}{2}\right) + \left(\frac{-60 - 90}{2}\right)\cos(2 \times -25) + 30\sin(-50) = -56.2\ (MPa)$$

$$\sigma_{y'} = \left(\frac{-60 + 90}{2}\right) + \left(\frac{-60 - 90}{2}\right)\cos(2 \times 65) + 30\sin(130) = 86.2\ (MPa)$$

$$\tau_{x'y'} = -\left(\frac{-60 - 90}{2}\right)\sin(-50) \ +\tau_{xy}\cos(-50) = -38.2\ (MPa)$$

PRINCIPAL (NORMAL) VALUES/PLANES

$$\sigma = \left(\frac{\sigma_x + \sigma_y}{2}\right) + \left(\frac{\sigma_x - \sigma_y}{2}\right)\cos(2\theta) + \tau_{xy}\sin(2\theta), \tau = -\left(\frac{\sigma_x - \sigma_y}{2}\right)\sin(2\theta) + \tau_{xy}\cos(2\theta)$$



$$\sigma_x = -60\ (MPa), \sigma_y = 90\ (MPa), \tau_{xy} = 30\ (MPa)$$

$$\tan(2\theta_P) = \frac{30}{\left(\frac{-60-90}{2}\right)} \Rightarrow 2\theta_P = -21.8\ or\ 180 + (-21.8) \Rightarrow \theta_p = -10.9\ or\ 79.1$$

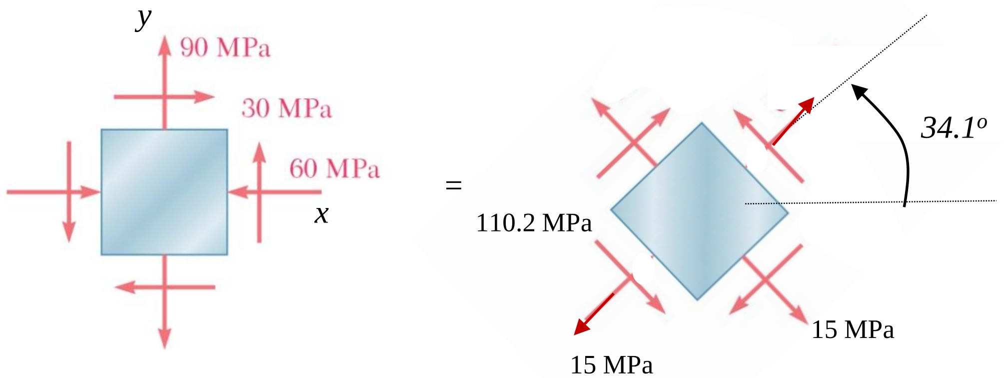
$$\sigma_1 = \sigma_{\theta=-10.9} = \left(\frac{-60+90}{2}\right) + \left(\frac{-60-90}{2}\right)\cos(2\times -10.9) + 30\sin(-21.8) = -95.2(MPa)$$

$$\sigma_2 = \sigma_{\theta=79.1} = \left(\frac{-60+90}{2}\right) + \left(\frac{-60-90}{2}\right)\cos(2\times 79.1) + 30\sin(158.2) = 125.2(MPa)$$

$$\tau_{\theta=-10.9,79.1} = -\left(\frac{-60-90}{2}\right)\sin(2\theta) + \tau_{xy}\cos(2\theta) = 0$$

PRINCIPAL SHEAR VALUES/PLANES

$$\sigma = \left(\frac{\sigma_x + \sigma_y}{2}\right) + \left(\frac{\sigma_x - \sigma_y}{2}\right)\cos(2\theta) + \tau_{xy}\sin(2\theta), \tau = -\left(\frac{\sigma_x - \sigma_y}{2}\right)\sin(2\theta) + \tau_{xy}\cos(2\theta)$$



$$\sigma_x = -60\ (MPa), \sigma_y = 90\ (MPa), \tau_{xy} = 30\ (MPa)$$

$$\tan(2\theta_s) = \frac{-\left(\frac{-60-90}{2}\right)}{30} \Rightarrow 2\theta_s = 68.2\ or\ 180 + (68.2) \Rightarrow \theta_p = 34.1\ or\ 124.1$$

$$\sigma_{\theta=34.1} = \left(\frac{-60+90}{2}\right) + \left(\frac{-60-90}{2}\right)\cos(2\times 34.1) + 30\sin(68.2) = 15\ (MPa)$$

$$\sigma_{\theta=124.1} = \left(\frac{-60+90}{2}\right) + \left(\frac{-60-90}{2}\right)\cos(2\times 124.1) + 30\sin(248.2) = 15\ (MPa)$$

$$\tau_{1/2} = \tau_{\theta=34.1/124.1} = -\left(\frac{-60-90}{2}\right)\sin(2\theta) + \tau_{xy}\cos(2\theta) = \pm 110.2\ (MPa)$$

The End